<u>The most efficient way to solve a Rubik's cube</u>

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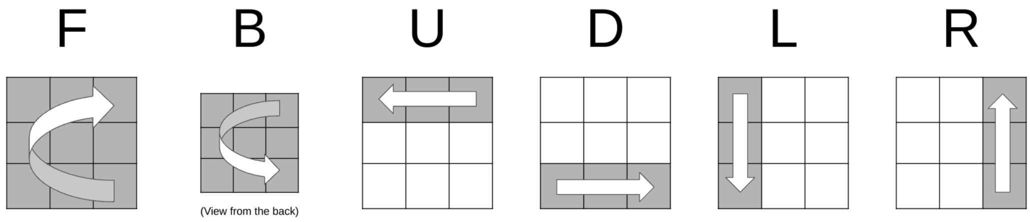
<u>A brief history and introduction</u>

The famous Rubik's cube is a 3D puzzle that was invented in 1974 by Hungarian sculptor and professor of architecture Ernő Rubik. Creating it in his mother's apartment, Ernő did not anticipate his idea would lead to him acquiring immense wealth and over 450 million cubes being sold worldwide. Rather fittingly, he commented, *'We turn the cube and it twists us'* as on the surface this relatively simplistic design is a toy to entertain children, however the complex understanding behind how this cube behaves has inspired mathematicians for decades.

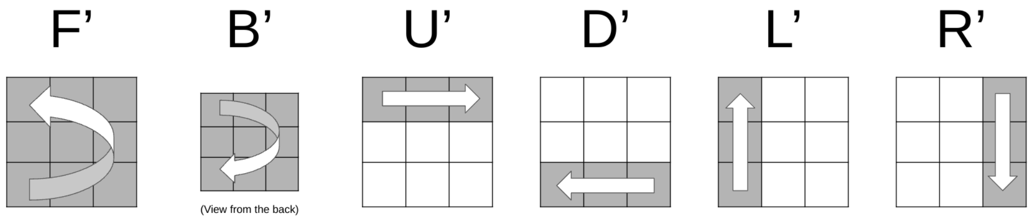
Many variations of the Rubik's cube with different shapes and dimensions have been made, however the one that will be discussed here is the standard 3 x 3 x 3 cube. Due to the pivot mechanism inside the cube which allows the faces to rotate independent of each other, the cube consists of 26 individual pieces (3 x 3 x 3 minus 1), which are commonly referred to as *'cubies'* or *'cubelets'*. The 6 faces of 9 panels each have a different colour and the objective is to return each face to a singular colour following what is known as a '*scramble*' of the cube. Initially, the colours of the faces varied from cube to cube but in 1988 it became standardised to have white opposite yellow, blue opposite green and red opposite orange.

Rubik's cube notation is used to describe the rotations of the cube faces and six letters will suffice solving the cube as a beginner. The combinations of letters and notation produce hundreds of algorithms which can be utilised at different stages of completing the cube to maximise the speed at which it is solved.

A single letter represents a 90 degree rotation of the corresponding cube face **clockwise**:



A single letter followed by an apostrophe represents a 90 degree rotation of the corresponding cube face **anticlockwise**:

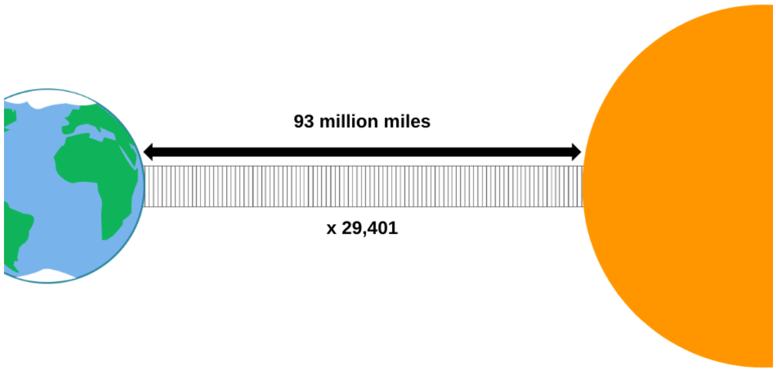


Often when presented with a Rubik's cube, a person's response to the question of how do you solve it is simply by removing the stickers and placing them back in their correct positions. The majority of modern day cubes produced however use plastic panels instead to prevent peeling and fading of colour. As a result, this method often isn't feasible and of course is very time consuming relative to the world record of 3.47 seconds!

So the question is, what is the most efficient way to solve a Rubik's cube? The answer to this question differs depending on the context. When solving a cube in real life, competitive speedcubers have hundreds of algorithms to choose from but shorter algorithms may not always equate to a shorter time as ergonomic factors come into play. Theoretically however, the meaning of efficiency here would simply be the shortest amount of moves possible, as this would naturally equate to the fastest solve time.

43 quintillion

The Rubik's cube has 43,252,003,274,489,856,000 possible permutations which is perhaps why randomly turning the cube to try and solve it is considerably unreasonable. To give you an idea of how big that number is, if you were to stack 43.252 quintillion pieces of paper on top of each other (paper thickness of 0.004 inches), the stack would reach from the earth to the sun, over 29,000 times!



Arriving at this huge value is actually relatively intuitive and entails exploring the possible positions for every corner piece and edge piece. This is because the centre pieces are fixed to the internal rotating mechanism so will always be in the same position on the cube.

Calculation:

Corners

Edges

$$\frac{(3^8 \, x \, 8!)(2^{12} \, x \, 12!)}{12}$$

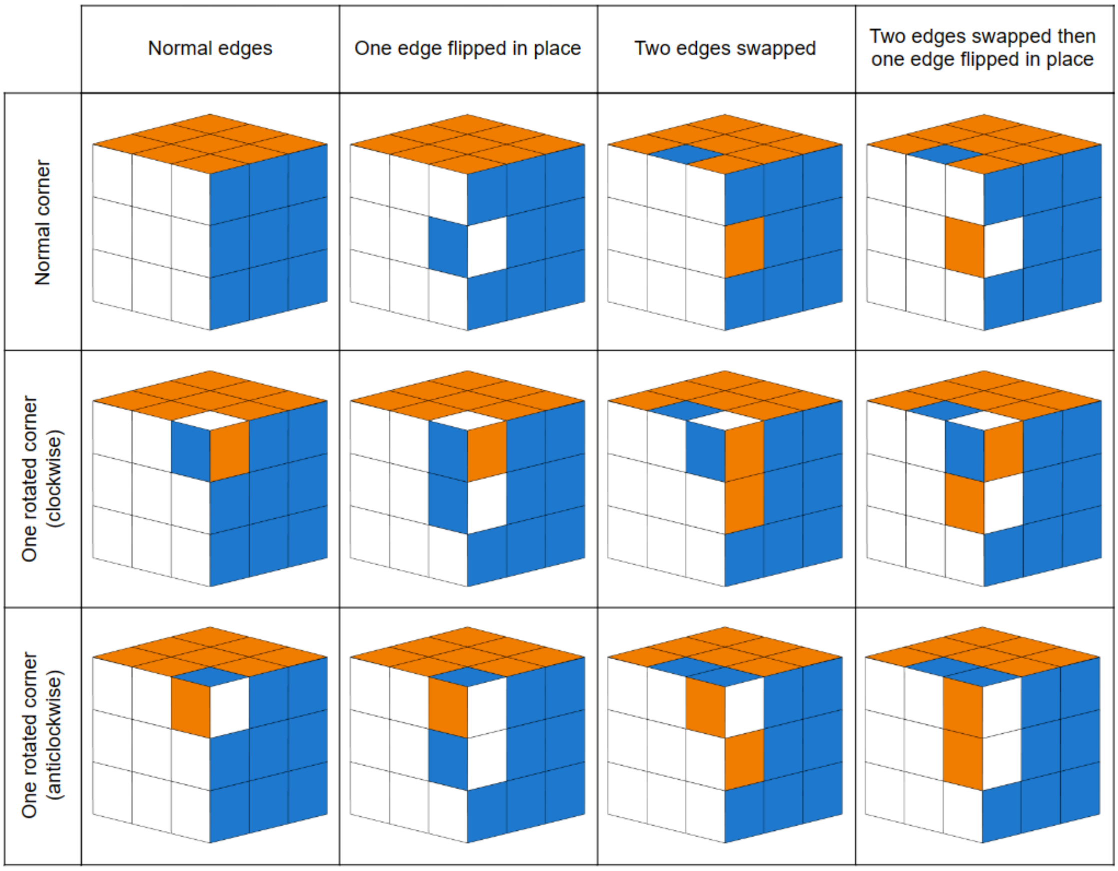
Accounting for solvable versions of the cube

Corners: If you envision a single corner piece of the cube it has 3 visible sides and therefore 3 possible orientations. This corner can be located in 8 different positions as a cube has 8 vertices. This means in each of the 8 positions there are 3 possible orientations which creates the first part of the calculation: 3^8

When assigning corners to create a permutation, the first corner has 8 '*choices*' in terms of where it is positioned. Once the first corner has been assigned, the second corner has 7 choices, the third corner 6 choices, the fourth corner 5, and so on. This leaves us with 8 factorial so now we have: $(3^8 \, x \, 8!)$

Edges: Applying the same process to the edges, a single edge piece has 2 possible orientations in 12 different locations: 2 to the power of 12. The first edge piece can be placed in 12 possible locations, the second edge in 11, the third 10, and so on. This leaves us with 12 factorial this time and now the top line of the calculation is complete: $(3^8 \, x \, 8!)(2^{12} \, x \, 12!)$

We have covered all the possible permutations of corners and edges on the cube now which leaves the question: where does the division of 12 come from? If you imagine disassembling and then reassembling a Rubik's cube, placing the pieces in any random order and orientation, the division of 12 would not apply and there would instead be 519 quintillion unique permutations of the cube! However, there would only be a 1/12th chance a given cube created by this method is actually solvable. Essentially, this is down to the fact that certain orientations of pieces, and therefore certain permutations of the cube, aren't solvable via an algorithm. The diagram below represents this visually:



There are no other arrangements of the cube that cannot be transformed into one of the 12 cubes above. The 12 actually comes from the multiplication of three factors: 3, 2 and 2.

The factor of 3 relates to the fact there is an algorithm to twist two corners, but there is no algorithm to twist a single corner. This means any number of corners ≥2 can be reduced down to one corner which has a ⅓ chance of being orientated correctly. This leads us back to the fact a single corner piece can be in one of 3 possible orientations.

Similarly, the first factor of 2 relates to the fact there is an algorithm to flip two edges in place, but there is no algorithm to flip a single edge in place. This means any number of edges ≥2 can be reduced down to one edge which has a ½ chance of being orientated correctly. Once again, this leads us back to the fact a single edge piece can be in one of two orientations.

The second factor of 2 can be explained using either corners or edges. In the diagram it is demonstrated with edges but the principle is the same for the corners. There is an algorithm which swaps two edges whilst simultaneously swapping two corners, but there is no algorithm which swaps a pair of edges only, nor a pair of corners only. Referring to the diagram, you can see by swapping the two edges located in the centre and upper left of the cube, you either jump from column 1 to column 3 or column 2 to column 4.

To summarise, there are 43 quintillion ways a Rubik’s cube could be scrambled and this number is derived from the combining of the corner and edge permutations, as well as accounting for the solvable versions of the cube (without taking it apart). This means there is a high likelihood that a well scrambled Rubik’s cube has never been scrambled in that way by anyone ever before!

Properties of the Rubik’s cube as a mathematical structure relating to group theory

A branch of algebra called group theory explains how the Rubik’s cube can be solved. It also explains the certain properties it possesses which must be considered when forming algorithms as the tricky thing about the cube is manipulating a piece into its desired position without affecting the rest of the cube. In maths, a group is defined as a set and a binary operator which satisfy four conditions known as group axioms: closure, associativity, identity and invertibility. In this context the cube permutations are the group elements and the sequences of rotations of the faces of the cube (better known as algorithms) act as binary operators. Notably, different move sequences which produce the same end result in terms of the cube’s arrangement are considered the same permutation. Let’s take a look at the group axioms applied to a Rubik’s cube:

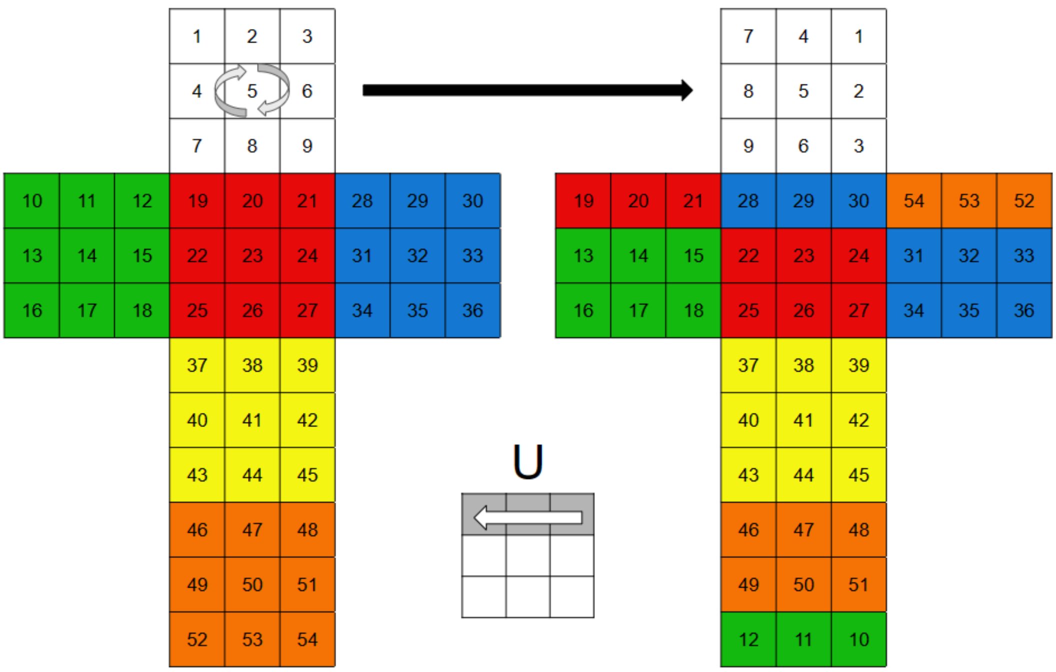
Closure: There is a predefined list of actions that never change. In this case the general rotations (F B U D L R F’ B’ U’ D’ L’ R’) are the actions.

Associativity: Parentheses can be placed anywhere in the operation and the result is equivalent to parentheses placed elsewhere in the same operation. In this case permutations can be grouped together:
E.g (F R’ U) L R = F R’ (U L R)

Identity (Neutral element): An operation which leaves the set unchanged or in a ‘*neutral’* state. In this case there are permutations which do not rearrange the set: E.g F F’ U U’ D’ D

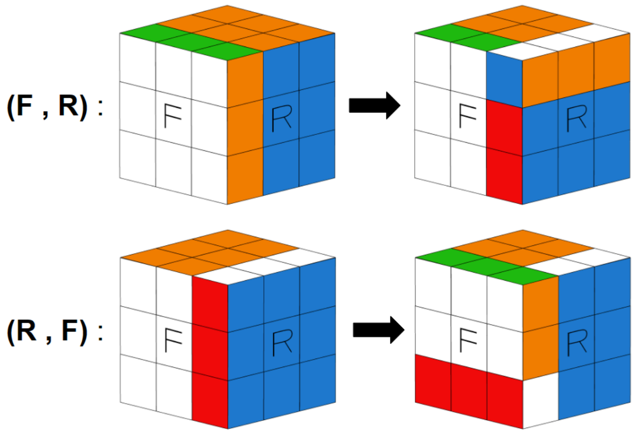
Invertibility (Inverse element): An operation which is the inverse of its initial operation. In this case every permutation has an inverse permutation: E.g R U F D has the inverse operation D’ F’ U’ R’

More specifically, a permutation group is a group whose elements are permutations of a finite, ordered list, and whose group operation is the permutations which rearrange the set in a certain way. Consequently, a Rubik’s cube is a permutation group as it can be represented as an ordered list of 54 elements (6 faces of different colours, consisting of 9 individual pieces on each face). The diagram below shows how the cube can be represented this way, and how just a single rotation of the upper face of the cube rearranges 20 out of the 54 elements:



Commutativity

Commutativity is where changing the order of the operands in an operation still gives the same result: E.g multiplication is commutative as $6 x 3 = 3 x 6$. Although not a required condition in group theory, commutativity is especially important to consider when forming algorithms for the cube. Some pairs of moves on the cube do commute: E.g F B = B F, and these move sequences are referred to as *'disjoint permutations'* as the faces operate independent of each due to the middle face separating them. The majority of pairs however do not commute, demonstrated visually in the diagram below. Notice how F R ≠ R F so F and R do not commute.



Using commutativity to form algorithms

Let X and Y be two moves on a Rubik's cube.

The commutator of X and Y is the sequence X Y X^{-1} Y^{-1}, which is denoted by [X , Y].

[X , Y] = 1 ⇔ XY = YX meaning the commutator of X and Y is equal to the identity if and only if X and Y commute.

Proof: [X , Y] = $(XY)(YX)^{-1}$, which equals the identity when $(YX)^{-1}$ is the inverse of XY, i.e. when XY = YX.

As a result, the commutator of X and Y measures the extent to which X and Y fail to commute. Using this non-commutativity, we can now see how algorithms are used to create a sequence of moves which results in moving or orientating the chosen pieces, and then essentially reversing the effects this has had on the rest of the cube faces. For example:

To rotate the top front left and top front right corners on the cube clockwise the commutator is:
[X , Y] = (F' D F L D L') U (L D' L' F' D' F) U' which can also be written as (to demonstrate the inverses),
[X , Y] = $(F^{-1}$ D F L D $L^{-1})$ U (L D^{-1} L^{-1} F^{-1} D^{-1} F) U^{-1} where:

X = F^{-1} D F L D L^{-1}
This rotates the top front left corner clockwise without disturbing the rest of the top layer. In this sequence of moves F^{-1} D F takes the corner off the top layer, and L D L^{-1} rotates the corner before returning it to the top layer. At this point the lower two layers of the cube are scrambled.

$Y = U$
This moves the top front right corner into the top front left position. As mentioned earlier, this single move is *'disjoint'* so does not affect the lower two layers of the cube.

$X^{-1} = L\ D^{-1}\ L^{-1}\ F^{-1}\ D^{-1}\ F$
This rotates the top front left corner, which was originally the front top right corner, counterclockwise and then unscrambles the lower two layers of the cube.

$Y^{-1} = U^{-1}$
Finally, this restores the top layer to its original position and the top front left and top front right corners on the cube have been rotated clockwise without the rest of the cube being disturbed.

God's number

A common misconception when someone is given a Rubik's cube to scramble for someone to solve is the longer time they spend scrambling or greater the number of moves they use, the harder it makes it to solve. Excluding scrambles which are only a few moves away from the solved cube of course, this is indeed false as it has been proven every single one of the 43,252,003,274,489,856,000 permutations can be solved in 20 moves or less. This remarkable fact is commonly referred to as '*God's number*' with the idea being a deity would solve a Rubik's cube in the least number of moves possible, the most efficient way possible.

The first attempt at finding God's number was not conclusive and estimated to be 277 however through advancements in technology and the method in which this number is calculated, this number has slowly decreased to a final value of 20 over several years. The first serious attempt was by Morwen Thistlethwaite, who published a method enabling the cube to be solved in no more than 52 moves. This method works by reducing the cube down into certain restricted positions, till the point where there is nowhere for the other pieces to go: E.g One of the steps in the method specifies only the moves F2, B2, U2, D2, L2, R2 are allowed (the 2 represents a 90 degree rotation clockwise twice, or a 'half turn' of the cube). This method was then developed into a more compact computer algorithm but the problem arose that to find God's number it would take 35 CPU years. Thankfully Google came to the rescue and supplied a large number of powerful computers to run the program and eventually calculate this number in only a few weeks in real time.

God's algorithm

What about dramatically increasing the efficiency at which someone can initially learn how to solve a Rubik's cube? What if you just had to learn one algorithm that when repeated would inevitably arrive at the solved version of the cube at some point? The good news is this algorithm does exist and is called *'God's algorithm'*, however the bad news is it might take a little while to learn.

Every single algorithm which can be performed on the cube has a degree of permutations which is the number of how many times the algorithm must be repeated to return the cube to its original position. Here of some examples:

Algorithm 1: F
This simple algorithm has a degree of 4, a length of 1 and so a total of 4 moves would be needed to return the cube to its original position when repeating this algorithm.

Algorithm 2: R U R' U'
This algorithm has a degree of 6, a length of 4 and so a total of 24 moves would be needed to return the cube to its original position when repeating this algorithm.

Algorithm 3: R U2 D' B D'
This algorithm has a degree of 1260, a length of 5 and so a total of 6300 moves would be needed to return the cube to its original position when repeating this algorithm. This particular degree of permutations (1260) has been proven to be the largest possible.

Any algorithm of degree D and length L can show at most DL different permutations when repeated any amount of times. For example, referring back to algorithm 2 above, R U R' U' shows 24 permutations only when repeated more than D times (in this D = case 6 times) as once the degree has been surpassed you are repeating the same 24 moves, forming a cycle. As we know the largest degree has been proven to be 1260, we can use this fact to work out how many moves God's algorithm would include.

For the algorithm to work, it must pass through all 43,252,003,274,489,856,000 permutations of the cube.

Therefore $DL \geq 43{,}252{,}003{,}274{,}489{,}856{,}000$

If $D = 1260$ then $1260L \geq 43{,}252{,}003{,}274{,}489{,}856{,}000$

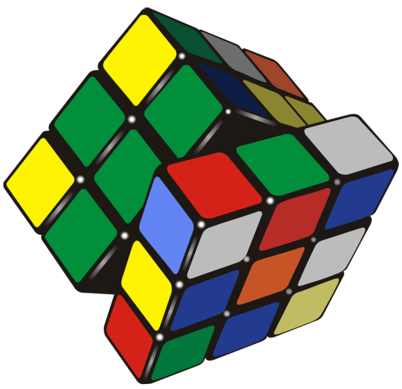
$L \geq 43{,}252{,}003{,}274{,}489{,}856{,}000/1260$

$L \geq 34{,}326{,}986{,}725{,}785{,}600$

So God's algorithm is **at least** 34,326,986,725,785,600 moves long! If you completed 100 moves a second it would take you approximately 10,885,016 years to get through this algorithm, which of course is not at all practical.

To conclude, if you are solving a Rubik's cube as a human being in the interest of speed, the shortest algorithms that are physically easiest to perform are of course optimal. If however you possess the mind to be able to test millions of permutations at once upon a glance of the cube, then your number of moves every solve will never exceed 20. The mathematics behind a Rubik's cube is extensive and rather extraordinary when comprehending the number of permutations for example. This simple cube composed of smaller simple cubes will continue to inspire mathematicians across the world for decades to come as they search for the still great quantity of unanswered questions. What exactly is God's number for a 4 x 4 x 4 perhaps? What is it for 5 x 5 x 5? Is there a pattern that emerges?

We continue to turn the cube and it will inevitably continue to twist us.



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